# Singularities in the Angular Momentum Plane

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Evidence for the existence of cuts in the angular momentum plane is discussed and the recent experimental data from CERN on $p$-$p$ scattering are interpreted in terms of a simple model with a Pomeranchuk pole $\alpha_p(t)$ $[\alpha_p(0)=1]$ and a cut extending as far as $\mathrm{Re}\,\alpha(t)=1$ for all $t$. A fit is obtained to the $p$-$p$ data in the region between 3–30 GeV/$c$ and there is a pronounced flattening of the curves for $d\sigma/dt$ above 20 GeV/$c$, but the total cross section still approaches a constant value and the model yields a shrinkage of the diffraction peak, though more slowly than in the single-pole model without a cut. The presence of a cut also provides a natural explanation of the effects previously attributed to a second Pomeranchuk trajectory. Due to the competition of the cut and the pole near $t=0$ the value of the trajectory $\alpha_p(t)$ cannot be accurately determined for small $|t|$ with the present experimental data, and it is possible that $d\alpha_p(0)/dt$ is, in fact, much smaller than that obtained from only a pole model and may even be zero. The calculations suggest that more accurate data with $t\sim-0.8$ are most likely to decide whether Regge poles are the only singularities determining high-energy scattering.

AN important experimental result in high-energy scattering has been achieved with the observation that the diffraction peak in elastic $p$-$p$ scattering shrinks with increasing energy.[1] This shrinkage appears to agree with the assertion that the high-energy behavior is dominated by Regge poles,[2–6] since the $p$-$p$ data from 3 to 30 GeV/$c$ fit the assumption of a single vacuum Regge pole trajectory. But it is by no means certain that the Regge poles are the only singularities present in the relativistic theory. It is interesting to consider other assumptions and see whether the high-energy data can be fitted by different "realistic" models, and whether future high-energy experiments can test the validity of any particular model based on conjectured analyticity properties of the $S$ matrix in the angular momentum plane. In this note particular attention is focused on the effects of including branch cuts in the angular momentum plane.

Under special conditions it has been shown for the relativistic $S$ matrix that $A(s,\alpha)$ is meromorphic in $\alpha$ for all $s$ and $\mathrm{Re}\,\alpha>1$.[7,8] This result depends upon the validity of the Mandelstam representation and upon a weak (two-body) form of unitarity; there are no conclusive results as yet on the analyticity properties of $A(s,\alpha)$ in the right-hand plane $\mathrm{Re}\,\alpha\geq 0$. Indeed, the optimistic potential analogy must be postulated as, for instance, in terms of the so-called principle of "maximum analyticity."[9] On the basis of an analogy to the Regge behavior in potential scattering, it has been suggested that cuts only occur when the scattering system gives rise to anomalous thresholds.[10] On the other hand, cuts are known to exist in the angular momentum plane for relativistic Coulomb scattering[11] and for the potential $-a/r^2$.[2] Moreover, Amati *et al.*[12] have derived a cut from their model field theory based on "multiperipheral" scattering by looking at the first multiple scattering iterate of the amplitude containing a Regge pole. The validity of their iteration scheme is somewhat suspect and unitarity is not satisfied in the $s$ channel, but we shall appeal to their results when deciding what type of branch cut to consider.

Our model of high-energy scattering is based on a dispersion relation in the angular momentum plane for the amplitude $A(t,\alpha)$. We use the familiar variables $s$, $t$, and $u$ for equal mass scalar particles with $s=4(q^2+M^2)$, $t=-2q^2(1-\cos\theta)$, and $u=-2q^2(1+\cos\theta)$, where $q$ is the momentum and $\theta$ the scattering angle in the c.m. system and $M$ is the mass of the particle. The partial wave amplitude $A(t,\alpha)$ is assumed to obey the dispersion relation, for fixed $t$, in the $\alpha$ plane:

$$A(t,\alpha)=\sum_i\frac{g_i(t)}{\alpha-\alpha_i}+\frac{1}{\pi}\int_C\frac{d\alpha'\,\rho(t,\alpha')}{\alpha'-\alpha},\qquad(1)$$

where the $g_i(t)$ are the residues corresponding to the Regge poles at $\alpha_i(t)$[13] and there is a cut along the contour $C$ with discontinuity $\rho(t,\alpha)$. We have assumed in (1) that $A(t,\alpha)\to 0$ as $|\alpha|\to\infty$. If $A(t,\alpha)$ is not sufficiently bounded, it can be divided by a polynomial in $\alpha(t)$.

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Consider the function

$$\Phi(t,\alpha)=A(t,\alpha)P_\alpha(-\cos\theta_t)[1\pm\exp(i\pi\alpha)]/\sin\pi\alpha, \quad (2)$$

which is analytic within the contour $\Gamma$ shown in Fig. 1, has poles at $\alpha_i(t)$ and at real positive integer values of $\alpha$. It is assumed that the cut in $A(t,\alpha)$ lies on the real axis between the branch points $\alpha_1$ and $\alpha_2$. We are not able at present to calculate the position or strength of such a cut and therefore restrict ourselves to the simplest assumption which can be tested by the high-energy data. From (2), we have

$$\frac{1}{2\pi i}\int_\Gamma \Phi(t,\alpha)d\alpha=0, \quad (3)$$

and it follows that

$$\begin{aligned}A(s,t)&=\textstyle\sum_l (-1)^l P_l(-\cos\theta_t)A_l(t)\\ &=\textstyle\sum_i g_i(t)P_{\alpha_i}(-\cos\theta_t)[1\pm\exp(i\pi\alpha_i)]/\sin\pi\alpha_i\\ &\quad+\frac{1}{\pi}\int_{\alpha_1}^{\alpha_2} d\alpha\, \mathrm{Im}A(t,\alpha)\\ &\quad\times P_\alpha(-\cos\theta_t)\frac{[1\pm\exp(i\pi\alpha)]}{\sin\pi\alpha}, \quad (4)\end{aligned}$$

where the reality condition $A^*(t,\alpha)=A(t,\alpha^*)$ has been used to identify $\rho(t,\alpha)$ with $\mathrm{Im}A(t,\alpha)$.

Three possibilities can be distinguished when the cut is included:

(a) The cut extends to $+\infty$. In this case there is no high-energy theory in the sense of Regge, since the discontinuity across the cut must be known to derive the asymptotic behavior.

(b) A cut extending as far as or farther to the right than the Regge pole, i.e., $\max[\mathrm{Re}\alpha{:}C]\geqslant\max[\mathrm{Re}\alpha_i]$.

(c) There is a pole farther to the right than the cut, i.e., $\max[\mathrm{Re}\alpha_i]>\max[\mathrm{Re}\alpha{:}C]$.

In case (a), the theory breaks down completely and in case (c) there will be a dominant pole and the cut can be neglected. In what follows, it is clearly most interesting to concentrate on case (b). The cut obtained by Amati *et al.*[12] appears to be of this type. With the conditions that $\alpha_p(0)=1$ and $d\alpha_p(0)/dt>0$ they find $\alpha_2(0)=1$ and $\alpha_2(t)\geq\alpha_p(t)$ for $t<0$, where $\alpha_p(t)$ is the position of the dominant pole and $\alpha_2(t)$ is the branch point farthest to the right. In the relativistic Coulomb problem and in the potential problem with $V=-a/r^2$ the branch cut depends only upon the coupling constant and not on the energy, e.g., in the Klein-Gordon case the cut occurs in the interval $-\frac{1}{2}-e^2\leq\mathrm{Re}\alpha\leq-\frac{1}{2}+e^2$.

We have $\cos\theta_t=(2s-2M^2+t)/(t-4M^2)$ with $s$ the invariant energy squared, and by virtue of $P_\alpha(\cos\theta_t)\approx(\cos\theta_t)^\alpha$ for large $\cos\theta_t$ and neglecting $t$ compared to $s$ and $M^2$, we get $(-\cos\theta_t)=(s-2M^2)/2M^2=E_L/M\approx s/2M^2$, where $E_L$ is the proton lab energy. In the

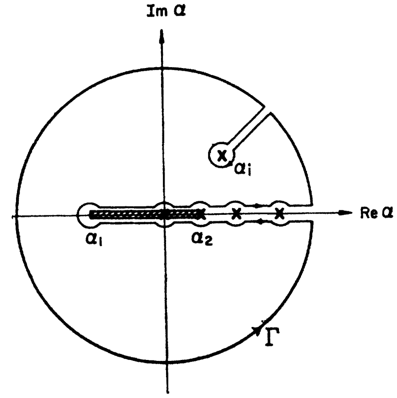


FIG. 1. The contour $\Gamma$ in the $\alpha$ plane showing positions of branch points $\alpha_1$ and $\alpha_2$ and the Regge pole $\alpha_i$.

high-energy limit $s\to\infty$ the cut contribution in (4) gives, by partial integration,

$$\begin{aligned}&\frac{1}{\pi}\int_{\alpha_1}^{\alpha_2} d\alpha\,\mathrm{Im}A(t,\alpha)\left(\frac{s}{2M^2}\right)^\alpha\frac{[1\pm\exp(i\pi\alpha)]}{\sin\pi\alpha}\\ &=\frac{1}{\pi}\left[\mathrm{Im}A(t,\alpha)\frac{(s/2M^2)^\alpha}{\ln(s/2M^2)}\frac{[1\pm\exp(i\pi\alpha)]}{\sin\pi\alpha}\right]_{\alpha_1}^{\alpha_2}\\ &\quad-\frac{1}{\pi}\int_{\alpha_1}^{\alpha_2} d\alpha\frac{d}{d\alpha}\left\{\mathrm{Im}A(t,\alpha)\frac{[1\pm\exp(i\pi\alpha)]}{\sin\pi\alpha}\right\}\frac{(s/2M^2)^\alpha}{\ln(s/2M^2)}\\ &\approx f(t)\frac{(s/2M^2)^{\alpha_2}}{\ln(s/2M^2)}\frac{[1\pm\exp(i\pi\alpha_2)]}{\sin\pi\alpha_2}, \quad (5)\end{aligned}$$

where we have put $f(t)=(1/\pi)\,\mathrm{Im}A(t,\alpha_2)$. In view of this result the asymptotic behavior of $A(s,t)$ resulting from (4) is

$$A(s,t)\approx\left[g_p(t)\left(\frac{s}{2M^2}\right)^{\alpha_p(t)}\epsilon_p+f(t)\frac{(s/2M^2)^{\alpha_2(t)}}{\ln(s/2M^2)}\epsilon_2\right], \quad (6)$$

where we write $\epsilon_i=[1\pm\exp(i\pi\alpha_i)]/\sin\pi\alpha_i$. We shall now fix $\alpha_2(t)=1$ in order to restrict ourselves to a definite model. This choice of $\alpha_2$ is not unreasonable in view of the work of Amati *et al.* and also in view of the knowledge that $\mathrm{Re}\alpha_p(t)\leq1$ for $t\leq0$.[14]

By virtue of the optical theorem we find that with $\alpha_2=1$ in (6) it follows that $\sigma_T\to$ const as $s\to\infty$. At high energies $A(s,t)$ is almost pure imaginary and the differential cross section is given by

$$\frac{d\sigma}{d\Omega}\propto s\left|g_p\left(\frac{s}{2M^2}\right)^{\alpha_p-1}+\frac{f}{\ln(s/2M^2)}\right|^2, \quad (7)$$

[14] The proof by Martin that $\mathrm{Re}\alpha_p(0)\leq1$ when only poles are present also applies to the amplitude including the cut, as given by Eq. (6), from which it can be shown that $\mathrm{Re}\alpha_p(0)\leq1$ and $\mathrm{Re}\alpha_2(0)\leq1$. A. Martin, in *Proceedings of the 1962 International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962), p. 566.

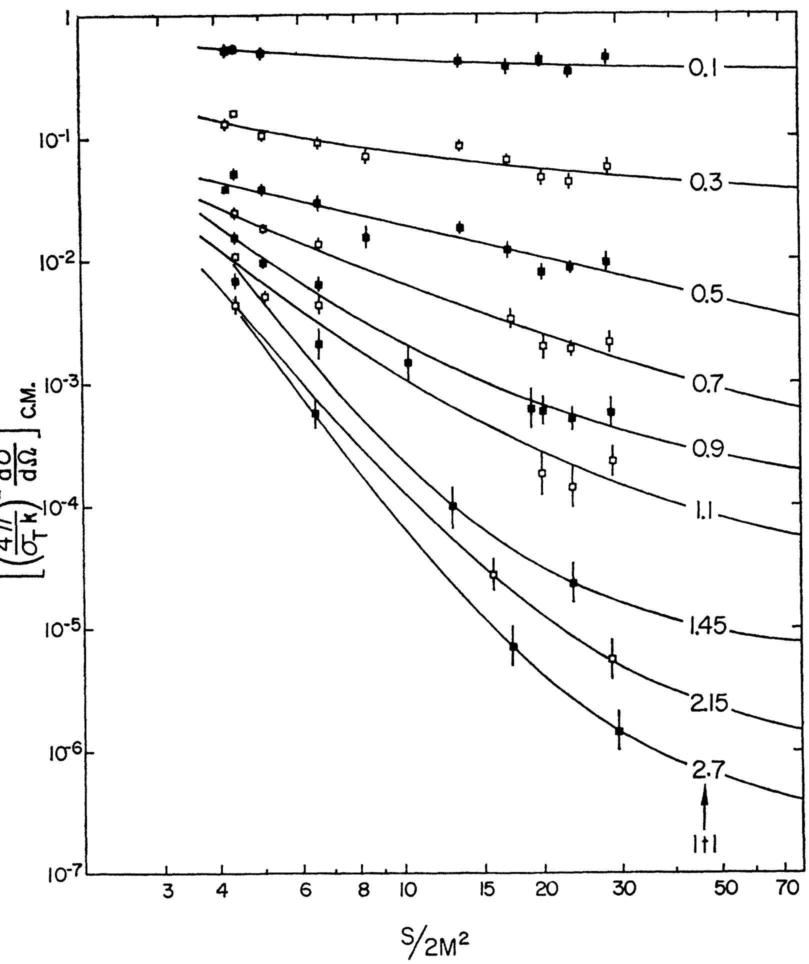


FIG. 2. The Pomeranchuk pole plus branch cut fit to the CERN experimental data for the $p$-$p$ elastic differential cross sections as a function of $s/2M^2$ (square of the c.m. total energy divided by twice the square of the nucleon mass). Appropriate values of $|t|$ are shown for each curve.

where it is assumed that $g_p$ and $f$ are constants. The diffraction peak can no longer be given exactly by an expression of the form $\exp(-\Gamma|t|)$, where $\Gamma$ is independent of $t$ if $g$ and $f$ are assumed constant, since the term $f/\ln s$ tends to extend the tail. However, if we define the width $\Gamma$ as the value of $1/|t|$ for which $(d\sigma/d\Omega)/(d\sigma/d\Omega)_{t=0}=1/e$, we find

$$\Gamma=2\alpha_p'(0)\ln\left(\frac{s}{2M^2}\right)\left[1-2\ln\left\{1-\frac{(e^{1/2}-1)f}{g\ln(s/2M^2)}\right\}\right]^{-1}$$
$$\rightarrow 2\alpha_p'(0)\ln(s/2M^2) \quad \text{as} \quad s\rightarrow\infty. \quad (8)$$

These results follow from the fact that for any given high energy the pole dominates the cut for very small angle scattering provided $\mathrm{Re}\alpha_2\leq 1$.

Let us now turn to the problem of $p$-$p$ and $\bar{p}$-$p$ elastic scattering. It is convenient to assume that the scattering at high energy is dominated by the spin-averaged amplitude. At least two poles must be considered corresponding to the Pomeranchuk and $\omega$ trajectories.[5,15] We adopt this two-pole approximation and similarly approximate the cut by two terms giving the contribution from the part of the cut farthest to the right. The asymptotic form of the amplitude is then

$$A(s,t)\approx\left[g_p(t)(s/2M^2)^{\alpha_p(t)}\epsilon_p\pm g_\omega(t)(s/2M^2)^{\alpha_\omega(t)}\epsilon_\omega\right.$$
$$\left.+f_c(t)\frac{(s/2M^2)^{\alpha_c(t)}}{\ln(s/2M^2)}\epsilon_c+f_2(t)\frac{(s/2M^2)^{\alpha_2(t)}}{\ln(s/2M^2)}\epsilon_2\right]. \quad (9)$$

In order to ensure the constancy of the $p$-$p$ total cross section above 10 GeV/$c$, we suppose that for $p$-$p$ scattering the $\omega$ and $c$ terms are approximately equal, so that in the $p$-$p$ amplitude their imaginary parts cancel for $t=0$ and $t<0$ and add in the $\bar{p}$-$p$ amplitude. Thus the third term in (9) plays the role of the "second" Pomeranchuk pole $P'$ with the same quantum numbers as the vacuum.[5,6,15]

To obtain a better understanding of the role played by a branch cut contribution, we have fitted the data on $p$-$p$ scattering reported by the CERN group[1] with an

[15] F. Hadjioannou, R. Phillips, and W. Rarita, Phys. Rev. Letters 9, 183 (1962).

equation of the form

$$\left(\frac{4\pi}{\sigma_T k}\right)^2 \frac{d\sigma}{d\Omega} = \left| g(t)\left(\frac{s}{2M^2}\right)^{\alpha_P(t)-1} + f(t)/\ln\left(\frac{s}{2M^2}\right)\right|^2, \quad (10)$$

assuming that the parameters $\alpha_p$, $g$, and $f$ are real. The values of $\alpha_p$, $g$, and $f$ as functions of $t$ for smaller values of $|t|$ are obtained by a least squares fit and for the larger values of $|t|$, where only three experimental points are known, by numerical solution of the simultaneous equations. The values of $\alpha_p$, $g$, and $f$ for each value of $t$ are given in Table I, and the corresponding curves are shown in Fig. 2 together with the experimental results. The dominance of the cut at high energies is apparent for larger values of $|t|$ in the flattening of the curves above $s/2M^2=20$, but data at higher energies will be necessary to establish the presence of a branch cut. For smaller values of $|t|$ the curves can hardly be distinguished experimentally from the straight lines given by the one-pole model, so that the calculation of $\alpha_p$, $g$, and $f$ for $|t|<0.7$ is not trustworthy, and we only consider seriously the curves for $|t|\geq 0.7$. This means that we cannot make predictions about the extrapolation of $\alpha_p(t)$ to positive values of $t$, but can only state that if a cut of the type treated here is present in the relativistic theory, then it is possible that $d\alpha/dt$ near $t=0$ may be small or even zero. The function $g(t)$ shows no regular features, but $f(t)$ for $|t|\geqslant 0.7$ is quite smooth. The discontinuity across the cut near $\alpha=1$ increases in magnitude as the pole $\alpha_p\to 1$ (i.e., as $t\to 0$) as would be expected if the cut were derived by an iterative procedure starting with a Pomeranchuk pole. The experimentally calculated values of $f(t)$ can be fitted well by an expression of the form

$$f(t)=[A+Bt^2+Ct^4]^{-1}, \quad (11)$$

with $A=0.542$, $B=21.4$, and $C=4.63$ if only those points for $|t|\geq 0.7$ are included, and with $A=2.412$, $B=16.3$, and $C=5.71$ if all points are included. In the above calculations, we have restricted $f$ to be positive. If this condition is relaxed one can obtain least squares solutions with $g>0$ and $f<0$. These solutions, in which the cut and pole compete, exhibit a zero of the total amplitude (assuming $f$ and $g$ real) and the solutions are probably unphysical.

TABLE I. Results of the experimental determination of $\alpha_p(t)$, $g(t)$, and $f(t)$ for fixed values of $t$ using the Pomeranchuk pole plus branch cut model with $\alpha_2(t)=1$.

| $t$ | $\alpha_p(t)$ | $g(t)$ | $f(t)$ |
|---|---|---|---|
| −0.1 | 0.99 | 0.56 | 0.25 |
| −0.3 | 0.95 | 0.15 | 0.33 |
| −0.5 | 0.56 | 0.39 | 0 |
| −0.7 | 0.11 | 0.40 | 0.072 |
| −0.9 | −0.64 | 0.95 | 0.058 |
| −1.1 | −0.58 | 0.83 | 0.029 |
| −1.45 | −1.61 | 0.43 | 0.012 |
| −2.15 | −1.23 | 1.69 | 0.0049 |
| −2.7 | −1.58 | 2.89 | 0.0026(5) |

To summarize, we reach the main conclusions:

(i) If the Pomeranchuk pole satisfies $\alpha_p(0)=1$, then it is possible for a cut to be present which extends as far to the right as $\mathrm{Re}\,\alpha=1$ and the diffraction peak will still exhibit a shrinkage and the total cross sections will asymptotically approach constants.

(ii) The combined pole and cut model gives a good fit to the $p$-$p$ data in the region between 3 and 30 GeV/$c$ and there is a definite flattening of the curves in Fig. 2 for wide-angle scattering above 20 GeV/$c$.

(iii) In our one-pole plus a branch cut model the value of $\alpha_p(t)$ cannot be determined accurately for small $t$, and consequently an extrapolation to positive $t$ is meaningless with the present data.

(iv) The best evidence for the existence of a cut comes from comparing the $p$-$p$ and $\bar{p}$-$p$ high-energy cross sections, since the branch cut provides a natural alternative to the conjectured second Pomeranchuk trajectory.